Sonoma State University — Physics 314
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SUMMARY OF SOME IMPORTANT CONCEPTS

I. MOMENTUM AND ENERGY – RELATIVISTIC AND NONRELATIVISTIC

A. Momentum is a conserved quantity. Its correct, relativistic definition is

$$\vec{\mathbf{p}} = \gamma m \vec{\mathbf{v}} \quad \text{where} \quad \gamma \equiv \frac{1}{\sqrt{1 - v^2/c^2}} \tag{1}$$

We may expand this in a power series in v^2/c^2:

$$\vec{\mathbf{p}} = m\vec{\mathbf{v}}\left[1 - v^2/c^2\right]^{-1/2} = m\vec{\mathbf{v}}\left[1 + \frac{1}{2}\left(\frac{v^2}{c^2}\right) + \ldots\right]$$

$$\vec{\mathbf{p}} \quad m\vec{\mathbf{v}} \quad \text{if } v^2/c^2 << 1 \qquad \textbf{nonrelativistic limit} \tag{2}$$

B. The **total energy** of a system is also a conserved quantity. Its correct, relativistic definition is

$$E = \gamma mc^2 + U \tag{3}$$

where $U = U(\vec{\mathbf{r}})$ is the potential energy and m is the rest mass. The energy of a particle may be further divided into three parts:

Energy	=	rest energy	+	kinetic energy	+	potential energy
E	=	E_0	+	K	+	$U(\vec{\mathbf{r}})$
	=	mc^2	+	$(\gamma - 1)mc^2$	+	$U(\vec{\mathbf{r}})$

where $E_0 \equiv mc^2$ is the **rest energy** of the particle (the energy it has in its rest frame in the absence of any potential energy), and $K = (\gamma - 1)mc^2$ is the **kinetic energy** of the particle. It is the energy the particle has due to its motion.

In the nonrelativistic limit, i.e., when $v^2/c^2 << 1$, the kinetic energy may be expanded in a power series:

$$K = (\gamma - 1)mc^2 = mc^2\left[\left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}} - 1\right] = mc^2\left[\frac{1}{2}\left(\frac{v^2}{c^2}\right) + \frac{3}{8}\left(\frac{v^2}{c^2}\right)^2 + \ldots\right]$$

$$K \quad \frac{1}{2}mv^2 \quad \text{if } v^2/c^2 << 1 \qquad \textbf{nonrelativistic limit} \tag{4}$$

C. The **energy-momentum relationship** is often useful, as energy and momentum are conserved quantities. Equations (1) and (3) may be combined to yield

$$E = \sqrt{\left(mc^2\right)^2 + \left(pc\right)^2} + U(\vec{\mathbf{r}}) \tag{5}$$

If $(p/mc)^2 << 1$, we may expand in a power series in $(p/mc)^2$:

$$E = mc^2\left[1 + \left(p/mc\right)^2\right]^{1/2} + U(\vec{\mathbf{r}}) = mc^2\left[1 + \frac{1}{2}\left(p/mc\right)^2 - \frac{1}{8}\left(p/mc\right)^4 + \cdots\right] + U(\vec{\mathbf{r}})$$

$$E \quad mc^2 + \frac{p^2}{2m} + U(\vec{\mathbf{r}}) \qquad \textbf{nonrelativistic limit}$$

It is customary to choose the arbitrary zero point of the potential energy so as to absorb the huge constant mc^2. Then we may write

$$E \quad \frac{p^2}{2m} + U(\vec{\mathbf{r}}) \quad \text{if} \quad (p/mc)^2 << 1 \qquad \textbf{nonrelativistic limit} \qquad (6)$$

This equation is the essence of Newtonian mechanics. It is equivalent to Newton's second law of motion. Another limit of the energy-momentum relation is useful when working with high energies. If $(p/mc)^2 >> 1$, we may expand in a power series in $(mc/p)^2$:

$$E = \left[\left(mc^2\right)^2 + (pc)^2\right]^{\frac{1}{2}} + V(\vec{\mathbf{r}}) = pc\left[1 + \left(\frac{mc}{p}\right)^2\right]^{\frac{1}{2}} + U(\vec{\mathbf{r}}) = pc\left[1 + \frac{1}{2}\left(\frac{mc}{p}\right)^2 + \cdots\right] + U(\vec{\mathbf{r}})$$

$$E \quad K \quad pc \quad \text{if} \quad (p/mc)^2 >> 1 \qquad \textbf{extreme relativistic limit} \qquad (7)$$

II. WAVE-PARTICLE DUALITY FOR LIGHT

A. Light as a wave: Many properties of electromagnetic radiation, e.g., interference, diffraction, dispersion, are best explained by describing the radiation as a wave phenomenon. As for all waves, velocity = frequency × wavelength:

$$v = \nu\lambda \qquad \textbf{all waves} \qquad (8)$$

The phase velocity of electromagnetic waves is $v = c/n$, where n, the index of refraction of the medium, is 1 for a vacuum, very slightly greater than 1 for air, and about 1.5 for glass. Thus in a vacuum,

$$c = \nu\lambda \qquad \textbf{electromagnetic waves} \qquad (9)$$

B. Light as a particle: Other properties of electromagnetic radiation, e.g., the blackbody spectrum, the photoelectric effect, and Compton scattering, can best be explained by describing the radiation as a stream of particles called **photons**. The photon may be considered a particle of rest mass zero. It obeys equation (5) but has no potential energy. Thus

$$E = K = pc \qquad \textbf{photons} \qquad (10)$$

In other words, the photon is always extremely relativistic.

C. The Planck-Einstein relation: The relationship between the wave and particle aspects of electromagnetic radiation is

$$E = h\nu \qquad \textbf{photons} \qquad (11)$$

Combining equations (10) and (11) we obtain

$$p = \frac{h}{\lambda} \qquad \textbf{photons} \qquad (12)$$

III. WAVE-PARTICLE DUALITY FOR PARTICLES OF NONZERO REST MASS

A. The de Broglie Relations: Equations (11) and (12) apply to all particles. Thus

$$v_{ph} = \nu\lambda \qquad \textbf{wave property} \qquad (13)$$

$$E = \sqrt{(pc)^2 + \left(mc^2\right)^2} + U \qquad \textbf{particle property} \qquad (14)$$

$$E = h\nu \qquad \textbf{wave-particle relation} \qquad (15)$$

$$p = \frac{h}{\lambda} \qquad \textbf{wave-particle relation} \qquad (16)$$

The above four equations apply to both “particles” and photons. We need only remember that $m = 0$ for photons.

B. Plane waves are sinusoidal in both space and time. An example is a wave moving in the positive x-direction:

$$\Psi(x,t) = Ae^{i(kx-\omega t)} \qquad \textbf{plane wave}$$

This wave has **wave number** $k = 2\pi/\lambda$ and **angular frequency** $\omega = 2\pi\nu$.

We can combine these with the deBroglie relations (15) and (16) to obtain

$$p = \hbar k \quad \text{and} \quad E = \hbar\omega \qquad (17)$$

A free particle has a wave function which is a plane wave.

IV THE SCHRÖDINGER EQUATION

A. Postulate that physical observables are to be replaced by **operators** which operate on the wave function $\Psi(x,t)$. Examples include

observable	**operator**
p_x	$\frac{\hbar}{i}\frac{\partial}{\partial x}$
x	x
E	$i\hbar\frac{\partial}{\partial t}$

(18)

Substitute these operators for their associated observables in equation (6) to obtain the **time-dependent Schrödinger equation** in one dimension:

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi(x,t) + U(x)\Psi(x,t) = i\hbar\frac{\partial}{\partial t}\Psi(x,t) \qquad (19)$$

B. In the common case where the potential is independent of time, we can separate the variables $\Psi(x,t) = \psi(x)f(t)$. It is then easy to show that

$$\Psi(x,t) = \psi(x)e^{-i\omega t} \qquad \text{where } \omega = \frac{E}{\hbar} \tag{20}$$

and $\psi(x)$ is the solution to the **time-independent Schrödinger equation:**

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) + U(x)\psi(x) = E\psi(x) \tag{21}$$

A satisfactory wave function must be single-valued and continuous, and its derivative $\frac{d\psi}{dx}$ must be continuous [except where $U(x) \to \infty$].

C. What to do with your wave function once you have it:

(1) Interpret it as a **probability amplitude**. Thus the probability that the particle is between x and $x + dx$ at time t is $P(x,t)\,dx$ where

$$P(x,t) = |\Psi(x,t)|^2 \tag{22}$$

If the time-dependent wave function is of the form of equation (20), then

$$P(x,t) = P(x) = |\psi(x)|^2 \tag{23}$$

(2) Find the **expectation value** of an observable by

$$\langle A \rangle = \int \Psi^*(x,t) A_{op} \Psi(x,t)\,dx \tag{24}$$

This is the average value to be expected from a number of measurements of identical systems. The wave function is an **eigenfunction** of operator A_{op} if

$$A_{op}\Psi(x,t) = a\Psi(x,t)$$

where a is a constant. Thus the wave function which satisfies the time-independent Schrödinger equation is an eigenfunction of the energy operator. The plane wave is an eigenfunction of the momentum operator since

$$p_{op}\Psi(x,t) = \frac{\hbar}{i}\frac{\partial}{\partial x}\Psi(x,t) = \frac{\hbar}{i}\frac{\partial}{\partial x}Ae^{i(kx-\omega t)} = \hbar k\,Ae^{i(kx-\omega t)} = \hbar k\Psi(x,t) \tag{25}$$

The constant a (equal to $\hbar k$ in the above example) is called an **eigenvalue.** It is the only value the observable can take in this quantum state.

D. Generalization to three dimensions:

The time-independent Schrödinger equation in three dimensions is

$$-\frac{\hbar^2}{2m}\nabla^2\psi(\vec{\mathbf{r}}) + U(\vec{\mathbf{r}})\psi(\vec{\mathbf{r}}) = E\psi(\vec{\mathbf{r}})$$

and the operators generalize in the obvious way, e.g.,

$$\vec{\mathbf{p}}_{op} = \frac{\hbar}{i}\vec{\nabla}$$