The Gamma Function

The gamma function appears frequently in physics problems. Some knowledge of its properties is very useful in evaluating integrals. The definition is

$$\boxed{\Gamma(z) \equiv \int_0^\infty e^{-t}\, t^{z-1}\, dt} \qquad (1)$$

Its most interesting property is obtained by integrating $\Gamma(z+1)$ by parts:

$$\Gamma(z+1) = \int_0^\infty e^{-t}\, t^z\, dt = \left[-e^{-t}\, t^z\right]_0^\infty - \int_0^\infty \left(-e^{-t}\right) z\, t^{z-1}\, dt \qquad \boxed{\Gamma(z+1) = z\Gamma(z)} \qquad (2)$$

We will show that $\Gamma(1) = 1$ and $\Gamma(1/2) = \sqrt{\pi}$. Then repeated application of (2) yields

$$\boxed{\Gamma(n+1) = n!} \qquad n = 1, 2, 3 \cdots \text{ and } \qquad \boxed{\Gamma\left(n+\tfrac{1}{2}\right) = \left(n-\tfrac{1}{2}\right)\left(n-\tfrac{3}{2}\right)\cdots\left(\tfrac{1}{2}\right)\sqrt{\pi}} \qquad (3)$$

[The fact that $\Gamma(1) = 1$ is the reason for the definition $0! = 1$].

Thus, for example, $\Gamma(6) = 5! = 5\ 4\ 3\ 2\ 1 = 120$ and $\Gamma\left(\frac{5}{2}\right) = \frac{3}{2}\,\frac{1}{2}\sqrt{\pi} = \frac{3\sqrt{\pi}}{4}$.

It is easy to show that $\Gamma(1) = \int_0^\infty e^{-t}\, dt = \left[-e^{-t}\right]_0^\infty = 1$.

To show that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ let $t = x^2$. Then $dt = 2x\, dx$ and

$$\Gamma\left(\tfrac{1}{2}\right) = \int_0^\infty e^{-t} t^{-1/2}\, dt = \int_0^\infty e^{-x^2} x^{-1}\, 2x\, dx = 2\int_0^\infty e^{-x^2}\, dx = \int_{-\infty}^\infty e^{-x^2}\, dx$$

Now multiply this integral by itself [using a different dummy variable of integration] and make it into a double integral:

$$\left[\Gamma\left(\tfrac{1}{2}\right)\right]^2 = \left[\int_{-\infty}^\infty e^{-x^2}\, dx\right]\left[\int_{-\infty}^\infty e^{-y^2}\, dy\right] = \iint e^{-\left(x^2+y^2\right)}\, dx\, dy$$

where the last expression is the integral over the entire x-y plane of the exponential of minus the square of the distance from the origin. Change to plane polar coordinates:
$x = r\cos\theta$, $y = r\sin\theta$, $dx\, dy = dA = r\, dr\, d\theta$.

$$\left[\Gamma\left(\tfrac{1}{2}\right)\right]^2 = \int_0^\infty e^{-r^2}\, r\, dr \int_0^{2\pi} d\theta = \left[-\tfrac{1}{2} e^{-r^2}\right]_0^\infty 2\pi = \pi \qquad \Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}$$

Now we can evaluate integrals of the form $\boxed{I_n(\lambda) \equiv \int_0^\infty e^{-\lambda x^2} x^n\, dx}$ by changing them to gamma functions. Let $t = \lambda x^2$, $x = (t/\lambda)^{1/2}$, $dx = \frac{1}{2}(\lambda t)^{-1/2}\, dt$.

$$I_n(\lambda) = \int_0^\infty e^{-t}\left(\frac{t}{\lambda}\right)^{\frac{n}{2}} \frac{1}{2}(\lambda t)^{-1/2}\, dt = \frac{1}{2}\lambda^{-(n+1)/2}\int_0^\infty e^{-t}\, t^{(n-1)/2}\, dt = \frac{1}{2}\lambda^{-(n+1)/2}\,\Gamma\left(\frac{n+1}{2}\right)$$

$$\boxed{I_0(\lambda) = \frac{1}{2}\sqrt{\frac{\pi}{\lambda}}, \quad I_1(\lambda) = \frac{1}{2\lambda}, \quad I_2(\lambda) = \frac{1}{4}\sqrt{\frac{\pi}{\lambda^3}}, \quad I_3(\lambda) = \frac{1}{2\lambda^2}, \quad I_4(\lambda) = \frac{3}{8}\sqrt{\frac{\pi}{\lambda^5}}, \text{ etc.}}$$

and, of course, $$\int_{-\infty}^\infty e^{-\lambda x^2} x^n\, dx = \begin{cases} 2I_n & \text{if } n = 0, 2, 4, \cdots \\ 0 & \text{if } n = 1, 3, 5, \cdots \end{cases}$$