

SUMMARY OF SOME IMPORTANT CONCEPTS

I. MOMENTUM AND ENERGY - RELATIVISTIC AND NONRELATIVISTIC

A. Momentum is a conserved quantity. Its correct, relativistic definition is

$$\vec{p} = \gamma m \vec{v} \quad \text{where} \quad \gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \quad (1)$$

We may expand this in a power series in v^2/c^2 :

$$\vec{p} = m \vec{v} [1 - v^2/c^2]^{-1/2} = m \vec{v} \left[1 + \frac{1}{2} \left(\frac{v^2}{c^2} \right) + \dots \right]$$

$$\vec{p} \approx m \vec{v} \quad \text{if} \quad v^2/c^2 \ll 1 \quad \text{nonrelativistic limit} \quad (2)$$

B. The total energy of a system is also a conserved quantity. Its correct, relativistic definition is

$$E = \gamma mc^2 + U \quad (3)$$

where $U = U(\vec{r})$ is the potential energy and m is the rest mass. The energy of a particle may be further divided into three parts:

$$\begin{aligned} \text{Energy} &= \text{rest energy} + \text{kinetic energy} + \text{potential energy} \\ E &= E_0 + K + U(\vec{r}) \\ &= mc^2 + (\gamma - 1)mc^2 + U(\vec{r}) \end{aligned}$$

where $E_0 = mc^2$ is the **rest energy** of the particle (the energy it has in its rest frame in the absence of any potential energy), and $K = (\gamma - 1)mc^2$ is the **kinetic energy** of the particle. It is the energy the particle has due to its motion.

In the nonrelativistic limit, i.e., when $v^2/c^2 \ll 1$, the kinetic energy may be expanded in a power series:

$$K = (\gamma - 1)mc^2 = mc^2 \left[\left(1 - \frac{v^2}{c^2} \right)^{-1/2} - 1 \right] = mc^2 \left[\frac{1}{2} \left(\frac{v^2}{c^2} \right) + \frac{3}{8} \left(\frac{v^2}{c^2} \right)^2 + \dots \right]$$

$$K \approx \frac{1}{2} mv^2 \quad \text{if} \quad v^2/c^2 \ll 1 \quad \text{nonrelativistic limit} \quad (4)$$

C. The energy-momentum relationship is often useful, as energy and momentum are conserved quantities. Equations (1) and (3) may be combined to yield

$$E = \sqrt{(mc^2)^2 + (pc)^2} + U(\vec{r}) \quad (5)$$

If $(p/mc)^2 \ll 1$, we may expand in a power series in $(p/mc)^2$:

$$E = mc^2 \left[1 + (p/mc)^2 \right]^{1/2} + U(\vec{r}) = mc^2 \left[1 + \frac{1}{2} (p/mc)^2 - \frac{1}{8} (p/mc)^4 + \dots \right] + U(\vec{r})$$

$$E \approx mc^2 + \frac{p^2}{2m} + U(\vec{r}) \quad \text{nonrelativistic limit}$$

It is customary to choose the arbitrary zero point of the potential energy so as to absorb the huge constant mc^2 . Then we may write

$$E = \frac{p^2}{2m} + U(\vec{r}) \quad \text{if } (p/mc)^2 \ll 1 \quad \text{nonrelativistic limit} \quad (6)$$

This equation is the essence of Newtonian mechanics. It is equivalent to Newton's second law of motion. Another limit of the energy-momentum relation is useful when working with high energies. If $(p/mc)^2 \gg 1$, we may expand in a power series in $(mc/p)^2$:

$$E = \left[(mc^2)^2 + (pc)^2 \right]^{\frac{1}{2}} + V(\vec{r}) = pc \left[1 + \left(\frac{mc}{p} \right)^2 \right]^{\frac{1}{2}} + U(\vec{r}) = pc \left[1 + \frac{1}{2} \left(\frac{mc}{p} \right)^2 + \dots \right] + U(\vec{r})$$

$$E = K = pc \quad \text{if } (p/mc)^2 \gg 1 \quad \text{extreme relativistic limit} \quad (7)$$

II. WAVE-PARTICLE DUALITY FOR LIGHT

A. Light as a wave: Many properties of electromagnetic radiation, e.g., interference, diffraction, dispersion, are best explained by describing the radiation as a wave phenomenon. As for all waves, velocity = frequency \times wavelength:

$$v = \nu \lambda \quad \text{all waves} \quad (8)$$

The phase velocity of electromagnetic waves is $v = c/n$, where n , the index of refraction of the medium, is 1 for a vacuum, very slightly greater than 1 for air, and about 1.5 for glass. Thus in a vacuum,

$$c = \nu \lambda \quad \text{electromagnetic waves} \quad (9)$$

B. Light as a particle: Other properties of electromagnetic radiation, e.g., the blackbody spectrum, the photoelectric effect, and Compton scattering, can best be explained by describing the radiation as a stream of particles called **photons**. The photon may be considered a particle of rest mass zero. It obeys equation (5) but has no potential energy. Thus

$$E = K = pc \quad \text{photons} \quad (10)$$

In other words, the photon is always extremely relativistic.

C. The Planck-Einstein relation: The relationship between the wave and particle aspects of electromagnetic radiation is

$$E = h\nu \quad \text{photons} \quad (11)$$

Combining equations (10) and (11) we obtain

$$p = \frac{h}{\lambda} \quad \text{photons} \quad (12)$$

III. WAVE-PARTICLE DUALITY FOR PARTICLES OF NONZERO REST MASS

A. The de Broglie Relations: Equations (11) and (12) apply to all particles. Thus

$$v_{ph} = v\lambda \quad \text{wave property} \quad (13)$$

$$E = \sqrt{(pc)^2 + (mc^2)^2} + U \quad \text{particle property} \quad (14)$$

$$E = h\nu \quad \text{wave-particle relation} \quad (15)$$

$$p = \frac{h}{\lambda} \quad \text{wave-particle relation} \quad (16)$$

The above four equations apply to both “particles” and photons. We need only remember that $m = 0$ for photons.

B. Plane waves are sinusoidal in both space and time. An example is a wave moving in the positive x -direction:

$$\Psi(x, t) = Ae^{i(kx - \omega t)} \quad \text{plane wave}$$

This wave has **wave number** $k = 2\pi/\lambda$ and **angular frequency** $\omega = 2\pi\nu$.

We can combine these with the deBroglie relations (15) and (16) to obtain

$$p = \hbar k \quad \text{and} \quad E = \hbar\omega \quad (17)$$

A free particle has a wave function which is a plane wave.

IV THE SCHRÖDINGER EQUATION

A. Postulate that physical observables are to be replaced by **operators** which operate on the wave function $\Psi(x, t)$. Examples include

observable	operator	
p_x	$\frac{\hbar}{i} \frac{\partial}{\partial x}$	(18)
x	x	
E	$i\hbar \frac{\partial}{\partial t}$	

Substitute these operators for their associated observables in equation (6) to obtain the **time-dependent Schrödinger equation** in one dimension:

$$-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \Psi(x, t) + U(x)\Psi(x, t) = i\hbar \frac{\partial}{\partial t} \Psi(x, t) \quad (19)$$

B. In the common case where the potential is independent of time, we can separate the variables $\Psi(x, t) = \psi(x)f(t)$. It is then easy to show that

$$\Psi(x, t) = \psi(x)e^{-i\omega t} \quad \text{where } \omega = \frac{E}{\hbar} \quad (20)$$

and $\psi(x)$ is the solution to the **time-independent Schrödinger equation**:

$$-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} \psi(x) + U(x)\psi(x) = E\psi(x) \quad (21)$$

A satisfactory wave function must be single-valued and continuous, and its derivative $\frac{d\psi}{dx}$ must be continuous [except where $U(x)$ is infinite].

C. What to do with your wave function once you have it:

(1) Interpret it as a **probability amplitude**. Thus the probability that the particle is between x and $x + dx$ at time t is $P(x, t) dx$ where

$$P(x, t) = |\Psi(x, t)|^2 \quad (22)$$

If the time-dependent wave function is of the form of equation (20), then

$$P(x, t) = P(x) = |\psi(x)|^2 \quad (23)$$

(2) Find the **expectation value** of an observable by

$$\langle A \rangle = \int \Psi^*(x, t) A_{op} \Psi(x, t) dx \quad (24)$$

This is the average value to be expected from a number of measurements of identical systems. The wave function is an **eigenfunction** of operator A_{op} if

$$A_{op} \Psi(x, t) = a \Psi(x, t)$$

where a is a constant. Thus the wave function which satisfies the time-independent Schrödinger equation is an eigenfunction of the energy operator. The plane wave is an eigenfunction of the momentum operator since

$$p_{op} \Psi(x, t) = \frac{\hbar}{i} \frac{d}{dx} \Psi(x, t) = \frac{\hbar}{i} \frac{d}{dx} A e^{i(kx - \omega t)} = \hbar k A e^{i(kx - \omega t)} = \hbar k \Psi(x, t) \quad (25)$$

The constant a (equal to $\hbar k$ in the above example) is called an **eigenvalue**. It is the only value the observable can take in this quantum state.

D. Generalization to three dimensions:

The time-independent Schrödinger equation in three dimensions is

$$-\frac{\hbar^2}{2m} \nabla^2 \psi(\mathbf{r}) + U(\mathbf{r})\psi(\mathbf{r}) = E\psi(\mathbf{r})$$

and the operators generalize in the obvious way, e.g.,

$$\mathbf{p}_{op} = \frac{\hbar}{i} \nabla$$